MAXIMAL COMMUTATIVE ALGEBRAS OF LINEAR TRANSFORMATIONS(1)

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The Problem; a Summary of the Main Results. In this paper K_n shall denote the algebra of n by n matrices over a field K or, equivalently, the algebra of linear transformations over an n-dimensional vector space. We attack the problem of determining conditions which imply that a commutative subalgebra of K_n is maximal commutative by seeking conditions on a commutative ring R with identity element and on a unital right R-module M satisfying the A.C.C. and the D.C.C. which imply:

$$(P_1) \qquad \mathrm{Hom}_R(M, M) = R^*$$

where R^* denotes the set of endomorphisms $\{a_r | x \rightarrow xa \text{ for } x \in M, a \in R\}$. The latter problem shall be referred to as the Centralizer Problem.

The following definition, due to E. Snapper [13, p. 125], is a prerequisite to one of the known results on the Centralizer Problem:

DEFINITION. An R-module M is said to be completely indecomposable provided that (1) R is a commutative ring with identity, (2) M satisfies the D.C.C. and the A.C.C., and (3) every submodule of M is indecomposable. (We prove in Theorem 1.5 that, if such a module is faithful, the chain conditions hold in R, also.)

An equivalent form [13, p. 127, Remark 1.2] of the preceding definition replaces (3) by (3'): M contains a unique irreducible submodule.

Two instances in which (P_1) holds are:

(P_2) M is a cyclic R-module for a commutative ring R with identity.

(P_3) M is a completely indecomposable module.

Let (P_2) hold and let $E \in \mathrm{Hom}_R(M, M)$. If M can be generated by $x \in M$ then for some $r \in R$, $E(x) = xr$ and it is easily seen that the mapping $m \rightarrow mr$, $m \in M$, is identical with E. The nontrivial result "$(P_3) \Rightarrow (P_1)$" is Snapper's [13, p. 129, Theorem 3.1]. The result intersects this paper only in a special instance, that in which M is a finite dimensional vector space over a field K and R is a commutative subalgebra of K_n containing the identity transformation. For this case the truth of Snapper's theorem is made apparent in §3 (Theorem 3.4 and Remark).

Presented to the Society, October 31, 1959; received by the editors February 10, 1960 and, in revised form, January 6, 1961.

(1) This paper forms part of a dissertation at the University of Wisconsin, written under the direction of Associate Professor Charles W. Curtis, to whom the author expresses sincerest appreciation.

I. Schur [12] proved that the maximal dimension of a maximal commutative subalgebra of K_n is $[n^2/4]+1$, provided K is an algebraically closed field. This result was generalized to an arbitrary field by N. Jacobson [8]. Also in [8], it is proved that if K is not an imperfect field of characteristic 2 then (up to conjugacy) exactly two algebras have the maximal dimension when n is odd and greater than one, and exactly one when n is even. Suprunenko [14] proved that if K is algebraically closed and if $n>6$ there exist infinitely many nonisomorphic maximal commutative subalgebras of K_n. Work on the problem has been published recently by B. Charles [3] and H. S. Allen [1]. A summary of the main results of this paper follows, beginning with the

NOTATION. If a function $(s_1, s_2)\rightarrow s_1s_2$ is defined on the Cartesian product of two sets S_1 and S_2 into a third set with zero element, Ann (S_1, S_2) shall denote $\{x \mid x\in S_2 \text{ and } (s_1, x)\rightarrow 0 \text{ for every } s_1\in S_1\}$. A similar definition holds for Ann (S_2, S_1).

In §1 the Centralizer Problem as stated above is reduced to the case in which M is an indecomposable module with chain conditions (Theorem 1.1). It follows (Theorem 1.5) that R/P is a field, where $P=\text{rad}(\text{Ann}(M, R))$, and that $MP^e=0$ for some integer $e>0$. With these restrictions on the ring, it is proved in the main theorem (§2, Theorem 2.7) that an R-module M satisfies (P_1) and is indecomposable provided $MP\neq 0$ and M/MP has a finite (R/P)-basis $x_1+MP, \cdots, x_k+MP$, $k>1$, such that for $i=1, \cdots, k$:

$$(C_1)\qquad \text{Ann}(P, M) \subseteq x_1P;$$

$$(C_2)\qquad P = \text{Ann}(x_i, R) + \bigcap_{s\neq i} \text{Ann}(x_s, R).$$

Because of the Chinese Remainder Condition (C_2), such a module is referred to as an R-module $[C]$. Example 4.1 in §4 is an R-module $[C]$ which does not satisfy (P_2) or (P_3) and Example 4.2 is an R-module $[C]$ which satisfies neither chain condition. It is an inference of Theorem 2.3 and motivation as well for the definition of R-module $[C]$, that M is one provided R/P is a field, MP is finitely generated and $\neq 0$, $1<(R/P)\text{-dim}\,(M/MP)<\infty$, $MP^2=0$, and (P_1) holds.

The application of (P_2), (P_3), and Theorem 2.7 to commutative algebras of linear transformations, with maximal radical P, acting on an n-dimensional vector space over a field K is considered in §§3 and 4. For the class of maximal commutative subalgebras of K_n so obtained it is shown in Theorem 3.6 that, if $R/P\cong K$, then dim $R-$dim $M=(k-1)(d-1)$, where k and d are the dimensions of M/MP and of $M/(\text{Ann}(P, M))$, respectively. Example 4.3, a maximal commutative subalgebra of K_9 with $R/P\cong K$, fails to satisfy Theorem 3.6, proving that the sufficient conditions considered here do not combine to make a necessary condition for (P_1).

1. Reduction of the problem.

THEOREM 1.1. *Let R be a commutative ring with identity and let M be an R-module which is a direct sum $M_1 \oplus \cdots \oplus M_k$ of k submodules. For $i=1, 2, \cdots, k$, let E_i denote the projection of M onto M_i. Then* (P_1) *is equivalent to*

(1.1) *for $i=1, 2, \cdots, k$, there is an element $e_i \in R$ such that $E_i(m)=me_i$ for all $m \in M$*

and

(1.2) $\operatorname{Hom}_R(M_i, M_i)=R^*=R_i^*=\operatorname{Hom}_{R_i}(M_i, M_i)$ *for each i, where R_j denotes e_jR $(1 \leqq j \leqq k)$ and where R^* is viewed as acting on M_i.*

Proof. From the definitions we have $\operatorname{Hom}_R(M_i, M_i) \subseteq \operatorname{Hom}_{R_i}(M_i, M_i)$ and $R_i^* \subseteq R^*$, and we prove the reverse inclusions, assuming (1.1). If $E \in \operatorname{Hom}_{R_i}(M_i, M_i)$ and $m \in M_i$ then $E(m) \in M_i$, so that $E(m)e_i=E(m)$. Now for $r \in R$, $E(mr)=E(me_ir)=E(m)e_ir=E(m)r$, proving $E \in \operatorname{Hom}_R(M_i, M_i)$. If $f \in R^*$, let $r \in R$ satisfy $f(m)=mr$, $m \in M_i$. From $m=me_i$ we have $f(m)=me_ir$ proving $f \in R_i^*$. Thus $R^*=R_i^*$ and $\operatorname{Hom}_{R_i}(M_i, M_i) =\operatorname{Hom}_R(M_i, M_i)$, assuming (1.1). The theorem is now reduced to the equivalence of (P_1) with (1.1) and

$$\operatorname{Hom}_R(M_i, M_i) = R^*. \tag{1.3}$$

Assuming (1.1) and (1.3), let $H \in \operatorname{Hom}_R(M, M)$. It is easily verified that $H_i[=H|_{M_i}] \in \operatorname{Hom}_R(M_i, M_i)$, $1 \leqq i \leqq k$. By (1.3) there exist elements $b_i \in R$, $1 \leqq i \leqq k$, such that $H_i(m)=mb_i$, $m \in M_i$. If $c=\sum_1^k e_ib_i$ and $m=\sum_1^k m_i$ is an element of M, then $H(m)=H(\sum m_i)=\sum H(m_i)=\sum H_i(m_i)=\sum H_i(m_ie_i) =\sum m_ie_ib_i=\sum me_ib_i=m\sum e_ib_i=mc$, proving ($P_1$). If ($P_1$) holds then (1.1) is immediate, since it is easily verified that $E_i \in \operatorname{Hom}_R(M, M)$. Let $G \in \operatorname{Hom}_R(M_i, M_i)$ and define H by $H(m)=G[E_i(m)]$, $m \in M$. Then $H=G \cdot E_i \in \operatorname{Hom}_R(M, M)$, and by ($P_1$) there is an element $h \in R$ such that $H(m)=mr$, $m \in M$. For $m \in M_i$, $mh=H(m)=G[E_i(m)]=G(m)$, proving (1.3). Q.E.D.

REMARK. If M satisfies the D.C.C., M is a direct sum of a finite number of indecomposable submodules [15, p. 169, Theorem 28]. The Centralizer Problem for such a module is reduced by Theorem 1.1 to the case in which M is indecomposable. The further reduction of the problem to modules for a completely primary, commutative ring assumes both chain conditions for M, and is the topic next investigated.

The terms prime ideal, primary ideal, and rad J for an ideal J shall have the conventional definitions as given in [11, pp. 9–13].

THEOREM 1.2. *If R is a commutative ring and M is an indecomposable R-module satisfying both chain conditions, then* $\operatorname{Ann}(M, R)$ *is a primary ideal and* $\operatorname{rad}(\operatorname{Ann}(M, R))$ *is a prime ideal.*

Proof. It is well known [6, p. 174] that the radical of a primary ideal is a prime ideal. To prove that $\text{Ann}(M, R)$ is primary, we observe first that, since the indecomposable module M satisfies the chain conditions, every non-nilpotent R-endomorphism of M is an automorphism (Fitting's Lemma [6, pp. 155–156]). If $Q = \text{Ann}(M, R)$ were not primary, there would exist in R elements s and t with $st \in Q$ and $s \notin Q$, $t^n \notin Q$, for all positive integers n. Now $s \notin Q$ implies that $Ms \neq 0$ and $st \in Q$ implies that the kernel of the map $E: m \to mt$, $m \in M$, contains the nonzero submodule Ms. Consequently, E is not an automorphism and must be nilpotent. But $t^n \notin Q$ for $n = 1, 2, \cdots$ contradicts nilpotency.

THEOREM 1.3. *If R is a commutative ring with identity, then the D.C.C. holds for ideals if, and only if,*

(1) *the A.C.C. holds for ideals;*

(2) *every prime ideal different from R is maximal.*

A proof of Theorem 1.3 appears in [15, p. 203, Theorem 2].

THEOREM 1.4. *If R is a commutative ring with identity and M is an R-module which satisfies the A.C.C., then $R/\text{Ann}(M, R)$ satisfies the A.C.C. for ideals.*

This result appears in [4, p. 245, Theorem 4]. The proof uses the fact that if $x \in R$ the module $R/\text{Ann}(x, R)$ is R-isomorphic to the submodule xR of M and hence satisfies the A.C.C. Then if $\{x_1, x_2, \cdots, x_n\}$ is a set of generators of M, $R/\text{Ann}(M, R) = R/(\bigcap_{i=1}^{n} \text{Ann}(x_i, R))$ satisfies the A.C.C. as a result of the theorem [4, p. 242, Theorem 3]: Let G be a group with operators R and let $G_1, G_2, \cdots, G_n$ be normal R-subgroups of G. If the A.C.C. holds for R-subgroups of G/G_i, $1 \leqq i \leqq n$, then the A.C.C. holds for R-subgroups of $G/(G_1 \cap \cdots \cap G_n)$.

DEFINITION. A commutative ring R is said to be completely primary provided that R contains an identity element and that R/J is a field, where J is the Jacobson radical of R. (If R satisfies the D.C.C. for ideals, $J = \text{rad } 0$ [9, pp. 38–39], so that the second requirement reads: $R/\text{rad } 0$ is a field. This form of the definition will be used when applicable without further comment.)

DEFINITION. Let R be a commutative ring and let J be an ideal of R. If M is an R-module, then the exponent of J relative to M is defined to be

$$\begin{cases} \text{the positive integer } e \text{ (if it exists) such that } J^e \text{ (but not } J^{e-1}) \subseteq \text{Ann}(M, R)\text{(2)}; \\ \infty, \text{ otherwise.} \end{cases}$$

THEOREM 1.5. *If M is a faithful indecomposable R-module having composition length $h < \infty$, where R is a commutative ring with identity, then*

(2) The convention $J^0 = R$ (for ideals J of R) is used.

(1) *R satisfies both chain conditions for ideals,*
(2) *R is completely primary,*
(3) $e \leqq h$ *where e is the exponent of* $P[=\text{rad } 0]$.

Proof. For $i=1, \cdots, h$ let $B_i=\{r \mid r\in R$ and $M_{i-1}r\subseteq M_i\}$, where $M=M_0\supset M_1\supset \cdots \supset M_h=\{0\}$ is a composition series of M. If $x\in M_{i-1}$, $x\notin M_i$, the irreducibility of M_{i-1}/M_i implies $(x+M_i)R=M_{i-1}/M_i$. Thus the mapping that takes r onto $(x+M_i)r$, $r\in R$, is an R-homomorphism of R onto M_{i-1}/M_i. Its kernel is $\{r \mid (x+M_i)r\subseteq M_i\}=\{r \mid M_{i-1}r\subseteq M_i\}=B_i$. The isomorphism of R/B_i with irreducible module M_{i-1}/M_i proves that B_i is a maximal proper ideal. Thus R/B_i is a field for $i=1, 2, \cdots, h$, so that the ideal P of nilpotent elements is contained in $\bigcap_{i=1}^h B_i$. Since M is indecomposable and satisfies both chain conditions by hypothesis, Theorem 1.2 asserts that P is a prime ideal. From the definition of the ideals B_i we see that $\prod_{i=1}^h B_i$ annihilates each element of M. Thus $\prod_{i=1}^h B_i\subseteq\{0\}\subseteq P$ and by the primality of P we have $B_i\subseteq P$ for some i. P, then, is maximal and from $P\subseteq B_i$ for all i we have $P=B_i(1\leqq i\leqq h)$. Thus $P^h=\prod_{i=1}^h B_i=0$, proving (3). By Theorem 1.4 R satisfies the A.C.C. for ideals. By Theorem 1.3 R will also satisfy the D.C.C. if every prime ideal different from R is maximal. If $Q[\neq R]$ is a prime ideal, then $Q=\text{rad } Q\supseteq\text{rad } 0=P$, so that Q is the maximal ideal P. Thus R satisfies the chain conditions, which is conclusion (1). Since rad 0 is a maximal ideal and R satisfies the D.C.C., R is completely primary.

THEOREM 1.6. *Let M be an R-module for a commutative ring R with identity such that* $\text{Hom}_R(M, M)=R^*$. *Assume also that R has a proper ideal P which contains every proper ideal containing* $\text{Ann}(M, R)$. *Then M is indecomposable.*

Proof. If M_1 and M_2 are nonzero submodules of M such that $M=M_1\oplus M_2$, we have for $i=1, 2$, $\text{Ann}(M, R)\subseteq\text{Ann}(M_i, R)\subset R$ (since $1\notin\text{Ann}(M_i, R)$), whence $P\supseteq\text{Ann}(M_i, R)$. If E is the projection of M onto M_1, $E\in\text{Hom}_R(M, M)$, so that by hypothesis there exists an element $b\in R$ such that $E(m)=mb$ for all $m\in M$. Clearly $0=M_1(1-b)=M_2b$. Thus the false statement

$$1 = (1-b)+b\in\text{Ann}(M_1, R)+\text{Ann}(M_2, R)\subseteq P$$

is implied, proving that M is indecomposable.

COROLLARY. *Let M be an R-module for a commutative ring R with identity such that* $P=\text{rad}(\text{Ann}(M, R))$ *is maximal in R. Then, if* $\text{Hom}_R(M, M)=R^*$, *M is indecomposable.*

Proof. If Q is a proper ideal containing $\text{Ann}(M, R)$, rad Q is a proper ideal since $1\notin\text{rad } Q$. Now from $P=\text{rad}(\text{Ann}(M, R))\subseteq\text{rad } Q$, P maximal, we have $P=\text{rad } Q$. Consequently, $Q\subseteq P$; P contains every ideal containing $\text{Ann}(M, R)$. Since the hypotheses of the theorem are satisfied, M is indecomposable.

DEFINITION. An element u of an R-module M is said to be a nongenerator of M if a subset S of M generates M whenever $S\cup\{u\}$ generates M.

REMARK. The set ϕ of nongenerators of M is the intersection of the maximal proper R-submodules ($\phi=M$ if there are no maximal proper submodules). This characterization of ϕ is proved for R-modules as it is proved for groups [10, p. 217]. Evidently ϕ is a submodule of M.

THEOREM 1.7. *Let $M[\neq 0]$ be an R-module for a commutative ring R with identity such that $P[=\text{rad}(\text{Ann}(M, R))]$ is a maximal ideal with finite exponent e. Then MP is the submodule ϕ of nongenerators of M.*

Proof. $M\neq MP$, for otherwise $M=MP=\cdots=MP^e=\{0\}$ contradicting $M\neq 0$. If $x\notin MP$ let $\{x_\alpha+MP\}$ for ordinals $\alpha=1, 2, \cdots$ be an (R/P)-basis of M/MP with $x=x_1$. If Y denotes $MP+\sum_{i>1}x_iR$, $x\notin Y$ (since the x_α are independent mod MP). Clearly the proper submodule Y is a maximal proper submodule. Since $x\notin Y$, x does not belong to the intersection ϕ of the maximal submodules. Thus the complement of ϕ contains the complement of MP. To prove $\phi\supseteq MP$ we need only prove $M_1\supseteq MP$ for every maximal proper submodule M_1. Since $M_1\not\supseteq M[=MR=MP^0]$ and $M_1\supseteq\{0\}=MP^e$, there exists a positive integer k such that $MP^k\not\subseteq M_1$ and $MP^{k+1}\subseteq M_1$. Let y be an element of MP^k, $y\notin M_1$. Since M/M_1 is irreducible $y+M_1$ generates M/M_1. If $x\notin M_1$, $x+M_1=(y+M_1)r$ for some $r\in R$; equivalently, $x\equiv yr$ modulo M_1. Then for all $p\in P$, $xp\equiv yrp$ modulo M_1 and, since $yrp\subseteq MP^{k+1}\subseteq M_1$, we have $xp\in M_1$. Now $xP\subseteq M_1$ for $x\notin M_1$ and $M_1P\subseteq M_1$ imply $MP\subseteq M_1$. Thus $MP\subseteq\phi$, $MP=\phi$.

COROLLARY. *As an additional hypothesis let MP be finitely generated; then* (1) $M=\sum_{i\in S}x_iR$ *and* (2) $MP=\sum_{i\in S}x_iP$ *where S is a set of indices and $\{x_i+MP\mid i\in S\}$ is an (R/P)-basis of M/MP. The (R/P)-dimension of M/MP will be finite (k, say) if M satisfies the A.C.C. and we will have* (3) $M=\sum_1^k x_iR$ *and* (4) $MP=\sum_1^k x_iP$.

Proof. Let $MP=\sum_1^m y_jR$. By the theorem the y_j are nongenerators of M and we conclude that $M=\sum x_iR+\sum_1^m y_jR=\sum x_iR+\sum_1^{m-1}y_jR=\cdots=\sum x_iR+y_1R=\sum x_iR$. If $z=\sum_{\alpha=1}^n x_{i_\alpha}r_{i_\alpha}\in MP$ then, by the independence mod MP of the x_{i_α}, $r_{i_\alpha}\in P$ for $\alpha=1, 2, \cdots, n$. Thus $MP\subseteq\sum x_iP$ and the reverse inequality is trivial. This completes the proof of (1) and (2).

If there is an infinite set of elements x_i, $i=1, 2, \cdots$, which are (R/P)-independent mod MP, a strictly ascending sequence $x_1R+MP\subset x_1R+x_2R+MP\subset\cdots$ is obtained; thus the assumption of the A.C.C. for M implies a finite dimension for M/MP. The A.C.C. also implies that MP is finitely generated, so that statements (1) and (2) hold and imply (3) and (4).

THEOREM 1.8. *Let M be a faithful R-module satisfying the A.C.C. for R-modules, where R is a commutative ring with identity. Let P be an ideal having* *finite exponent e such that R/P is a field. Then M is cyclic if, and only if, the (R/P)-dimension of M/MP is unity.*

Proof. Since M satisfies the A.C.C., the (R/P)-dimension of M/MP is finite by the corollary to Theorem 1.7. If $k=(R/P)\text{-dim } M/MP$ and $x_1, \cdots, x_k$ are (R/P)-independent mod MP, then by the corollary to Theorem 1.7 $M=\sum_1^k x_iR$; if $k=1$, $M=x_1R$. If, conversely, $M=xR$ for some $x\in M$, the mapping $r\rightarrow xr$, $r\in R$, is an isomorphism of R onto M, since M is faithful. Since by hypothesis P is a maximal ideal of R, the isomorphic image xP of P is a maximal submodule of M. As in the proof of Theorem 1.7 we have $MP\neq M$ from $e<\infty$, and from $xP\subseteq MP\subset M$ we conclude that MP is the maximal submodule xP. By the R-isomorphism $r\rightarrow xr$, $r\in R$, $\dim M/MP = \dim M/xP = \dim (R/P) = 1$.

2. **General results on the Centralizer Problem.** Investigation of $\text{Hom}_R(M, M)=R^*$, where R is a commutative ring with identity and M is an indecomposable R-module with both chain conditions, replaces the problem for the more general module M with chain conditions (Theorem 1.1 and Remark). It is clear that the radical P of $\text{Ann}(M, R)$ is a maximal ideal since, by Theorem 1.5, P is maximal among the ideals containing $\text{Ann}(M, R)$. By the same theorem the exponent e of P is finite, and $R/\text{Ann}(M, R)$ satisfies both chain conditions. Since M satisfies the A.C.C. the (R/P)-space M/MP has finite dimension (corollary to Theorem 1.7). The A.C.C. for M also implies that the (R/P)-space $\text{Ann}(P, M)$ has finite dimension and that MP is finitely generated.

The following lemma holds for an arbitrary ring R of operators. Its proof is straightforward.

LEMMA 2.1. *Let a group M with operators R contain a set of elements $\{x_1, \cdots, x_k\}$ such that $M=\sum_1^k x_iR$. For a fixed $y\in M$ and $j\in\{1, \cdots, k\}$ let E be the mapping defined by $E(\sum_1^k x_ir_i)=yr_j$. Then, if E is well-defined, $E\in\text{Hom}_R(M, M)$.*

LEMMA 2.2. *Let R be a commutative ring with identity element and maximal ideal P. Let M be an indecomposable R-module such that $\text{Ann}(M, R)\subseteq P$ and $MP\neq 0$. Assume that $M=\sum x_iR$ holds for any set $\{x_i\}$ such that $\{x_i+MP\}$ is an (R/P)-basis of M/MP. Then $\text{Ann}(P, M)\subseteq MP$.*

Proof. The result is obvious if $MP=M$. Let $MP\neq M$ and suppose the conclusion false. If $x\notin MP$ is an element of $\text{Ann}(P, M)$, let $x+MP$ be the first element of a basis $\{x_i+MP\}$ of M/MP. If $\dim(M/MP)=1$ we have $M=xR$ by hypothesis. Then $MP=xRP\subseteq xP\subseteq \text{Ann}(P, M)\cdot P=\{0\}$, contrary to hypothesis. If $\dim(M/MP)>1$, x_1R and $\sum_{i>1} x_iR$ are nonzero submodules. If $0=\sum x_ib_i$, $b_i\in R$, then, by the independence of the x_i, $b_i\in P$ for each i. Consequently, $x_1b_1\in\text{Ann}(P, M)\cdot P=\{0\}$ and $\sum_{i>1} x_ib_i=0$. Thus the sum $x_1R+\sum_{i>1} x_iR$ is direct, contrary to hypothesis. The contradictions prove $\text{Ann}(P, M)\subseteq MP$.

THEOREM 2.3. *Let R be a commutative ring with identity and let M be an R-module satisfying* $\text{Hom}_R(M, M)=R^*$. *Let P denote* $\text{rad}(\text{Ann}(M, R))$ *and assume further that R/P is a field, that P has finite exponent $e>1$, that MP is finitely generated, and that the dimension k of the (R/P)-space M/MP is finite. Then the following conclusions hold:*

(1) *M is indecomposable;*

(2) $\text{Ann}(P, M)\subseteq MP$;

(3) *if* $x\notin MP$, $\text{Ann}(P, M)\subseteq xP$;

(4) $\text{Ann}(MP, R)\subseteq \text{Ann}(x_i, R)+\bigcap_{j\neq i}\text{Ann}(x_j, R)$, $1\leqq i\leqq k$, *provided* $k\geqq 2$, *for any set* $\{x_1, \cdots, x_k\}$ *of k elements (R/P)-independent* mod MP;

(4') *using the notation of* (4), $P=\text{Ann}(x_i, R)+\bigcap_{j\neq i}\text{Ann}(x_j, R)$, $1\leqq i\leqq k$, *provided* $e=2$ *and* $k\geqq 2$;

(5) $(R/P)\text{-dim}[\text{Ann}(MP, R)/\text{Ann}(M, R)] = kd$, *where* $d = (R/P)\text{-dim}$ $\text{Ann}(P, M)$ *is assumed finite.*

Proof. Since $\text{Hom}_R(M, M)=R^*$ and $\text{rad}(\text{Ann}(M, R))$ is maximal in R, the corollary to Theorem 1.6 asserts the indecomposability of M, which is conclusion (1). MP is assumed to be finitely generated and e is assumed finite in order that $M=\sum_1^k x_iR$ hold for any set $\{x_1, \cdots, x_k\}$ of k elements (R/P)-independent mod MP (corollary to Theorem 1.7). We also have $MP\neq 0$ from $e>1$ and the conclusion of Lemma 2.2, $\text{Ann}(P, M)\subseteq MP$, follows. Thus conclusion (2) is proved.

If (3) is false, let $x\notin MP$ and $y\in\text{Ann}(P, M)$ be such that $y\notin xP$. Let $x=x_1, x_2, \cdots, x_k$ be k elements (R/P)-independent mod MP. Let E be the mapping defined by $E(\sum_1^k x_ib_i)=yb_1$, $b_i\in R$. Then E is well defined; for if $0=\sum x_ib_i$ then, by the independence of the x_i, $b_i\in P$ for each i and $yb_1 \in\text{Ann}(P, M)\cdot P=\{0\}$. By Lemma 2.1 $E\in\text{Hom}_R(M, M)$ and there is by hypothesis an element $h\in R$ such that $E(m)=mh$ for all $m\in M$. If $h\notin P$ the coset $h+P$ has an inverse $g+P$ for some $g\in R$; thus $gh=1+t$ for some $t\in P$. From

$$x_1 + x_1t = x_1gh = E(x_1g) = yg \in \text{Ann}(P, M) \subseteq MP$$

and $x_1t\in MP$, we have $x_1\in MP$ contrary to the choice of x_1. We have $h\in P$, whence $y=E(x_1)=x_1h\in x_1P$, contrary to the choice of y and $x_1[=x]$, which proves (3).

In the remainder of the proof $\{x_1, \cdots, x_k\}$ denotes a set of k elements independent mod MP.

Toward proving (4) let $i\in\{1, 2, \cdots, k\}$ and let $q\in\text{Ann}(MP, R)$. Let E be the mapping defined by

$$E\left(\sum_1^k x_jb_j\right) = x_ib_iq, \qquad b_j \in R.$$

E, then, is well-defined since E takes $0[=\sum x_i b_i]$ onto an element $x_i b_i q \in MP\cdot\text{Ann}(MP, R)=\{0\}$. By Lemma 2.1, $E\in\text{Hom}_R(M, M)$ and by hypothesis there exists an element $h\in R$ such that $E(m)=mh$ for all $m\in M$. In particular, $x_i h = E(x_i) = x_i q$ and $x_j h = E(x_j) = 0$ if $j\neq i$ or, equivalently, $h\in\bigcap_{j\neq i}\text{Ann}(x_j, R)$ and $q-h\in\text{Ann}(x_i, R)$. From $q=(q-h)+h$, we have (4).

If $e=2$, $P\subseteq\text{Ann}(MP, R)$ and we have $P\subseteq\text{Ann}(x_i, R)+\bigcap_{s\neq i}\text{Ann}(x_s, R)$ $(1\leqq i\leqq k)$ by conclusion (4). The reverse inequality holds, since the (R/P)-independence of the x_s implies $\text{Ann}(x_s, R)\subseteq P$, $s=1, \cdots, k$, proving (4′).

If $y_1, y_2, \cdots, y_d$ are d (R/P)-independent elements of $\text{Ann}(P, M)$ and if kd mappings E_{ij}, $1\leqq i\leqq k$, $1\leqq j\leqq d$, are defined by

$$E_{ij}\left(\sum_{s=1}^{k} x_s b_s\right) = y_j b_i, \qquad b_s\in R$$

the E_{ij} take $0[=\sum x_s b_s]$ onto $y_j b_i\in\text{Ann}(P, M)\cdot P=\{0\}$. By Lemma 2.1 $E_{ij}\in\text{Hom}_R(M, M)$, and by hypothesis there exist kd elements $u_{ij}\in R$ such that $E_{ij}(m)=mu_{ij}$ for all $m\in M$. We observe that $Mu_{ij}\subseteq\sum_1^d y_i R=\text{Ann}(P, M)$. Now for $m\in M$ and $p\in P$, $mpu_{ij}\in\text{Ann}(P, M)\cdot P=\{0\}$, so that $u_{ij}\in\text{Ann}(MP, R)$ for all i and j. To prove the independence of the elements u_{ij} we suppose that $\sum_{i=1}^k\sum_{j=1}^d u_{ij}r_{ij}=0$ with the $r_{ij}\in R$. Then, for fixed s, $x_s(\sum_{i,j}u_{ij}r_{ij})=0$ implies $x_s(\sum_{j=1}^d u_{sj}r_{sj})=\sum_{j=1}^d E_{sj}(x_s)r_{sj}=\sum_{j=1}^d y_j r_{sj}=0$, whence $\{r_{s1}, \cdots, r_{sd}\}\subseteq P$ by the independence of the y_j. This proves the (R/P)-independence of the u_{ij}. To prove that the kd element u_{ij} generate $\text{Ann}(MP, R)$—modulo $\text{Ann}(M, R)$—we let $h\in\text{Ann}(MP, R)$ and, for $i=1, 2, \cdots, k$, have $x_i h\cdot P=\{0\}$ or, equivalently, $x_i h\in\text{Ann}(P, M)$. Thus $x_i h=\sum_{j=1}^d y_j t_{ij}$ with $t_{ij}\in R$. For any element $\sum_{s=1}^k x_s b_s$ of M we have:

$$\sum_{s=1}^{k} x_s b_s h = \sum_{i,j} y_j b_i t_{ij} = \sum_{i,j}(x_i b_i u_{ij})t_{ij} = \sum_{s=1}^{k}(x_s b_s)\left(\sum_{i,j} t_{ij}u_{ij}\right)$$

whence $h-\sum t_{ij}u_{ij}\in\text{Ann}(M, R)$. Thus the (R/P)-dimension of $\text{Ann}(MP, R)/\text{Ann}(M, R)$ is kd. Q.E.D.

Corollary. *If R is a commutative ring with identity and $M[\neq 0]$ is an indecomposable module satisfying both chain conditions and* $\text{Hom}_R(M, M)=R^*$, *then either the conclusions* (2) *through* (5) *of the theorem hold or* $\text{Ann}(M, R)$ *is its own radical P. If $P=\text{Ann}(M, R)$, M is a 1-dimensional (R/P)-space.*

Proof. Except for $e>1$ the hypotheses of Theorem 2.3 were deduced from the hypotheses of this corollary in §1 and listed as opening observations of §2. Thus we have conclusions (2) through (5) or $e=1$ (equivalently, $P=\text{Ann}(M, R)$). A vector space over a field $F[\cong(R/P)]$ with dimension $\geqq 2$ has nonscalar linear transformations, that is, transformations not in F^*. Thus the (R/P)-dimension of $M[\neq 0]$ is unity in the case $P=\text{Ann}(M, R)$.

According to conclusion (4′) of Theorem 2.3, a necessary condition for $\text{Hom}_R(M, M)=R^*$ (when it is assumed that $e=2$, $k\geqq 2$) is that P satisfy a

Chinese Remainder Condition with respect to a set of ideals. For $k \geqq 2$ and arbitrary exponent $e>1$ conclusions (3) and (4′) combine to make a sufficient condition. To facilitate the discussion we make the

DEFINITION OF R-MODULE $[C]$. Let M be an R-module for a commutative ring R with identity such that R/P is a field where $P=\operatorname{rad}(\operatorname{Ann}(M, R))$. M shall be called an R-module $[C]$—for Chinese Remainder Condition—provided the exponent e of P and the (R/P)-dimension k of M/MP are finite and greater than one; and provided further that there exist k elements $x_1, \cdots, x_k$ (R/P)-independent modulo MP such that $M=\sum_1^k x_iR$ and that for $i=1, 2, \cdots, k$

(C_1) $\operatorname{Ann}(P, M) \subseteq x_iP$;

(C_2) $P=\operatorname{Ann}(x_i, R)+\bigcap_{s\neq i}\operatorname{Ann}(x_s, R)$.

The existence of such modules is demonstrated by examples (Examples 4.1 and 4.2) in §4. The module of Example 4.1 is neither cyclic nor completely indecomposable, so that the class of modules satisfying $\operatorname{Hom}_R(M, M)=R^*$ is properly extended by the main theorem (Theorem 2.7), which establishes that property for R-modules $[C]$. The notation of the definition will be assumed without further comment wherever R-modules $[C]$ arise. We shall also use equivalent forms of (C_2) as stated in the

LEMMA 2.4. (C_2) *in the definition of R-module* $[C]$ *is equivalent to each of*

(C_2') *for* $p\in P$, $i\in\{1, 2, \cdots, k\}$, *there is an element* $p'\in P$ *such that* $p\equiv p' \bmod \operatorname{Ann}(x_i, R)$ *and* $p'\equiv 0 \bmod \bigcap_{s\neq i}\operatorname{Ann}(x_s, R)$;

(C_2'') *if, for* $i=1, \cdots, k$, $b_i\in P$, *the k equations* $x_ib=x_ib_i$ *have a solution* $b\in P$.

Proof. Let $i\in\{1, \cdots, k\}$ and $p\in P$ and first assume (C_2''). Setting $b_i=p$ and $b_s=0$ for $s\neq i$ we obtain an element b such that $x_ib=x_ip$ (equivalently $p-b\in\operatorname{Ann}(x_i, R)$) and $x_sb=0$ if $s\neq i$ (equivalently $b\in\bigcap_{s\neq i}\operatorname{Ann}(x_s, R)$). Since $p=(p-b)+b$ we have (C_2). Assuming (C_2), i fixed, and $p\in P$ there exist elements $q'\in\operatorname{Ann}(x_i, R)$ and $p'\in\bigcap_{s\neq i}\operatorname{Ann}(x_s, R)$ such that $p=q'+p'$. Since $x_i(p-p')=x_iq'=0$, $p\equiv p' \bmod \operatorname{Ann}(x_i, R)$, so that p' satisfies the requirement of (C_2'). If (C_2') is assumed and $b_i\in P$ for $1\leqq i\leqq k$, then elements q_i exist such that $b_i\equiv q_i \bmod \operatorname{Ann}(x_i, R)$ for each i, while, for $s\neq i$, $x_sq_i=0$. Then, if $b=\sum_1^k q_i$, we have for each j: $x_jb=\sum_{i=1}^k x_jq_i=x_jq_j=x_jb_j$, proving ($C_2''$).

LEMMA 2.5. *If M is a R-module* $[C]$, *then* $x_iP\neq 0$ *for* $i=1, \cdots, k$.

Proof. By the definition of exponent $MP^{e-1}\neq 0$. From (C_1) and $MP^e=0$ we have $x_iP\supseteq\operatorname{Ann}(P, M)\supseteq MP^{e-1}$.

LEMMA 2.6. *If M is an R-module* $[C]$ *then, for* $j=1, \cdots, k$, $x_jP=\operatorname{Ann}(\operatorname{Ann}(x_j, R), MP)$.

Proof. The integer j is a constant in the proof. If $h\in\operatorname{Ann}(\operatorname{Ann}(x_j, R), MP)$ we write $h=\sum_1^k x_sn_s$ with $n_s\in P$ (since $MP=\sum_1^k x_iP$). Let $i\neq j$ be a fixed integer and let $p \in P$. There exists an element $p' \in P$ satisfying $x_i p = x_i p'$ and for $s \neq i$ $x_s p' = 0$ (Lemma 2.4, (C_2'')); in particular $x_j p' = 0$. By definition of h, $hp' = 0$ and we have

$$0 = hp' = \sum_{s=1}^{k} x_s n_s p' = x_i n_i p' = x_i n_i p.$$

For $i \neq j$ we have proved $x_i n_i \subseteq \mathrm{Ann}(P, M)$. Combining this with ($C_1$) we have $x_i n_i \in x_j P$ for $i \neq j$ (obviously for $i = j$), so that $h \in x_j P$. We have $\mathrm{Ann}(\mathrm{Ann}(x_j, R), MP) \subseteq x_j P$ and the reverse inequality is trivial.

THEOREM 2.7. *If M is an R-module* $[C]$, $\mathrm{Hom}_R(M, M) = R^*$ *and M is indecomposable.*

Proof. If $E \in \mathrm{Hom}_R(M, M)$ we take advantage of $M = \sum_1^k x_i R$ and write $E(x_i) = \sum_1^k x_j a_{ij}$, $a_{ij} \in R$, $i = 1, \cdots, k$. We claim that

$$\text{for } i \neq j, \qquad a_{ji} \in P. \tag{2.1}$$

If i and $j \neq i$ are fixed integers there exists by Lemma 2.5 an element $q \in P$ such that $x_i q \neq 0$. By (C_2'), Lemma 2.4, q may be chosen so that $x_s q = 0$ if $s \neq i$. In particular $x_j q = 0$ and we have:

$$0 = E(x_j q) = \left(\sum_{s=1}^{k} x_s a_{js} \right) q = x_i a_{ji} q. \tag{2.2}$$

If $a_{ji} \notin P$ there is an element b such that $a_{ji} b \in 1 + P$; thus $a_{ji} b = 1 + f$, $f \in P$. From (2.2) follows $x_i q + x_i q f = x_i q a_{ji} b = 0$, whence $x_i q = -x_i q f$. Now the choice of q is contradicted by $0 = (-1)^e x_i q f^e = -x_i q f = x_i q$; we have $a_{ji} \in P$ as claimed. Next we prove

$$\text{for } i \in \{2, 3, \cdots, k\}, \qquad a_{ii} - a_{11} \in P. \tag{2.3}$$

By Lemma 2.5 and (C_2'), Lemma 2.4, there is an element $q \in P$ such that $x_i q$ is a nonzero element of $\mathrm{Ann}(P, M)$ and $x_s q = 0$ if $s \neq i$. Since $x_i q \in \mathrm{Ann}(P, M)$ $\subseteq x_1 P$ by (C_1), there is an element $t \in P$ such that $x_i q = -x_1 t$ and we choose t so that $x_s t = 0$ if $s \neq 1$. We have

$$\begin{aligned} 0 = E(0) = E(x_i q + x_1 t) &= \sum_{j=1}^{k} x_j a_{ij} q \\ &+ \sum_{j=1}^{k} x_j a_{1j} t = x_i a_{ii} q + x_1 a_{11} t \\ &= x_i (a_{ii} - a_{11}) q. \end{aligned} \tag{2.4}$$

If $a_{ii} - a_{11} \notin P$, let $b \in R$ and $f \in P$ satisfy $(a_{ii} - a_{11}) b = 1 + f$. From (2.4) we have $x_i q + x_i q f = x_i (a_{ii} - a_{11}) q b = 0$, whence $x_i q = -x_i q f = (-1)^e x_i q f^e = 0$. Since the choice of q has been contradicted, $a_{ii} - a_{11} \in P$, proving (2.3).

We denote a_{11} by a and $x_i(a_{ii}-a)+\sum_{j\neq i} x_j a_{ij}$ by z_i, so that $E(x_i)=\sum x_j a_{ij} = x_i a + z_i$, $i=1, \cdots, k$. As a consequence of (2.1) and (2.3) $z_i \in MP$. Defining T by $T(m)=E(m)-ma$ for $m\in M$ we have $T(x_i)=z_i\in MP$; also,

$$E(m) = ma + T(m) \tag{2.5}$$

and

$$T \in \text{Hom}_R(M, M). \tag{2.6}$$

If $f\in \text{Ann}(x_i, R)$, $0=T(0)=T(x_i f)=z_i f$ (by (2.6)), so that $z_i=T(x_i) \in \text{Ann}(\text{Ann}(x_i, R), MP)$ which, by Lemma 2.6, equals $x_i P$. Consequently, there exist elements $s_i\in P$ such that $T(x_i)=x_i s_i$ for $i=1, 2, \cdots, k$. Taking advantage of (C_2''), Lemma 2.4, there is an element $s\in P$ such that $x_i s = x_i s_i$ and we have $T(x_i)=x_i s_i = x_i s$, $i=1, 2, \cdots, k$. Since $M=\sum_1^k x_i R$, $T(m)=ms$ for $m\in M$ and $E(m)=m(a+s)$ by (2.5). Thus $E\in R^*$.

Since P is a maximal ideal and $\text{Hom}_R(M, M)=R^*$, the indecomposability of M is implied by the corollary to Theorem 1.6.

In view of the omission of the chain conditions in the hypotheses of Theorem 2.7 it appears that the theorem has wider application than to the stated problem, which concerns direct summands of modules with chain conditions. Example 4.2 in §4 is an R-module $[C]$ which satisfies neither the A.C.C. nor the D.C.C.

THEOREM 2.8. *If M is an R-module $[C]$ and if $\bar{x}$ denotes the coset $x+\text{Ann}(P, M)$, then*
(i) $M/\text{Ann}(P, M)=\bar{x}_1 R\oplus \cdots \oplus \bar{x}_k R$;
(ii) $MP/\text{Ann}(P, M)=\bar{x}_1 P\oplus \cdots \oplus \bar{x}_k P$.

Proof. From the definition of R-module $[C]$ $M=\sum_1^k x_i R$; $MP=\sum_1^k x_i P$ is deduced as in the proof of the corollary, Theorem 1.7. Consequently (i) and (ii) will follow if we prove for each i

$$x_i P \cap \left(\sum_{s\neq i} x_s P\right) = \text{Ann}(P, M) \tag{2.7}$$

and

$$x_i R \cap \left(\sum_{s\neq i} x_s R\right) = \text{Ann}(P, M). \tag{2.8}$$

The identities

$$\text{Ann}(J_1+J_2, M) = \text{Ann}(J_1, M)\cap \text{Ann}(J_2, M),$$
$$\text{Ann}(J_1\cap J_2, M) \supseteq \text{Ann}(J_1, M)+\text{Ann}(J_2, M)$$

(which hold for ideals J_1 and J_2), applied to (C_2), yield

$$\begin{aligned}\text{Ann}(P, M) &= [\text{Ann}(\text{Ann}(x_i, R), M)] \cap \left[\text{Ann}\left(\bigcap_{s\neq i} (\text{Ann}(x_s, R)), M\right)\right] \\ &\supseteq [\text{Ann}(\text{Ann}(x_i, R), M)] \cap \left[\left(\sum_{s\neq i} \text{Ann}(\text{Ann}(x_s, R), M\right)\right].\end{aligned} \tag{2.9}$$

From (C_1) and $MP = \sum_1^k x_i P$ we have $\text{Ann}(P, M) \subseteq MP$, so that

$$\text{Ann}(P, M) = \text{Ann}(P, M) \cap MP. \tag{2.10}$$

Applying (2.10) to (2.9), we have

$$\begin{aligned}\text{Ann}(P, M) &\supseteq [\text{Ann}(\text{Ann}(x_i, R), M) \cap MP] \\ &\cap \left[\left(\sum_{s\neq i} \text{Ann}(\text{Ann}(x_s, R), M)\right) \cap MP\right] \\ &\supseteq \text{Ann}(\text{Ann}(x_i, R), MP) \\ &\cap \left[\sum_{s\neq i} (\text{Ann}(\text{Ann}(x_s, R), MP))\right].\end{aligned} \tag{2.11}$$

By Lemma 2.6 $\text{Ann}(\text{Ann}(x_j, R), MP)$ may be replaced in (2.11) by x_jP, $1 \leqq j \leqq k$, giving

$$\text{Ann}(P, M) \supseteq x_iP \cap \left(\sum_{s\neq i} x_sP\right).$$

The reverse inequality follows from $x_jP \supseteq \text{Ann}(P, M)$, $1 \leqq j \leqq k$, and (2.7) is proved. From the independence mod MP of the x_j follows $x_iR \cap (\sum_{s\neq i} x_sR) \subseteq MP$. Thus if $x_ib_i = \sum_{s\neq i} x_sb_s$ is an element of $x_iR \cap (\sum_{s\neq i} x_sR)$, we have $b_s \in P$ for $s = 1, 2, \cdots, k$. It follows that the left member of (2.8) is a subset of the left member of (2.7) and the reverse inequality between left members of (2.7) and (2.8) is trivial. (2.8) is proved by comparison of its left and right members with those of (2.7).

3. **Maximal commutative, completely primary algebras.** In the examples in §4 the module M is a vector space and R is a ring of linear transformations. Some theorems prerequisite to these examples is the concern of §3. Let K be a field and let K_n denote the full set of n by n matrices over K or, equivalently, the algebra $L(M, M)$ of linear transformations of an n-dimensional vector space M into itself. Let R be a subalgebra of K_n containing the identity matrix I_n. The n-dimensional space on which K_n acts shall be referred to as the representation space of K_n and of R.

Let M' be a vector space such that M and M' are dual according to a nondegenerate bilinear form [7, pp. 140–141]:

$$(m, m') \to g(m, m'), \qquad m \in M, \qquad m' \in M', \qquad g(m, m') \in K.$$

We shall write (m, m') for $g(m, m')$. Evidently M' and M have the same dimension [7, p. 141]. By the definition of nondegenerate [7, p. 140] we have

$$(3.1) \qquad (m, m') = 0 \text{ for all } m \in M \text{ implies } m' = 0;$$

$$(3.2) \qquad (m, m') = 0 \text{ for all } m' \in M' \text{ implies } m = 0.$$

If $r \in R$ and $m' \in M'$ the mapping $m \to (mr, m')$ is a linear functional; consequently, [7, p. 141] there exists a unique element $v' \in M'$ such that $(m, v') = (mr, m')$. It can be verified that the map $m' \to v'$ determined in this way by r is a linear transformation of M' into itself. More briefly, we say that to each $r \in R$

$$(3.3) \qquad (m, m'r') = (mr, m'), \qquad m \in M, m' \in M'$$

determines a linear transformation r' (the adjoint of r) which acts on M'. Let r_1 and r_2 be elements of R such that, for all $m \in M$ and $m' \in M'$, $(mr_1, m') = (mr_2, m')$. Then by (3.2) $mr_1 = mr_2$ for each m, and $r_1 = r_2$ since M is faithful. Thus the map $r \to r'$ is one-to-one. It is well known that R and $R' = \{r' \mid r \in R$ and (3.3) define $r'\}$ are anti-isomorphic rings, that M' is a right R'-module, and (assuming dual bases for M and M') that the matrix form of r' is the transpose of that of r. P and P' shall stand respectively for rad R and rad R'. It is also evident from the anti-isomorphism that P and P' have the same exponent, which we denote by e.

Except in Theorems 3.1 and 3.2 commutativity is assumed for R and the anti-isomorphism is an isomorphism. Except as stated specifically R/P is not assumed to be a field. R is assumed to possess the identity transformation I_n throughout §3.

THEOREM 3.1. (i) *The ring R is commutative [maximal commutative] if, and only if, the ring R' of adjoints is so*; (ii) *if M_1 is an R-module*, $\text{Ann}(M_1, M')$ *is an R'-module*; (iii) *if M_1 and M_2 are submodules of M such that $M = M_1 \oplus M_2$, then* $M' = \text{Ann}(M_1, M') \oplus \text{Ann}(M_2, M')$.

Proof. (i) is deduced from the anti-isomorphism of R with R' and (iii) holds for submodules since it holds more generally for subspaces [5, p. 31]. To obtain (ii) we let $r' \in R'$ and $x' \in \text{Ann}(M_1, M')$; then for all $m \in M_1$ we have $(m, x'r') = (mr, x') = 0$, since $mr \in M_1$. Thus $x'r' \in \text{Ann}(M_1, M')$. Q.E.D.

From the anti-isomorphism it is clear that an element $p \in R$ is nilpotent if, and only if, its adjoint p' is nilpotent. If p is a product of s nilpotent elements $0 < s \leqq e$, then by the anti-isomorphism the adjoint element p' is also; similarly for sums of such products. It follows that for $s = 0, 1, \cdots, e$ $p \in P^s$ if, and only if, its adjoint element $p' \in P'^s$(3).

THEOREM 3.2. *Let M, M', R, R', P and P' meet the general specifications of this section; for $s = 0, 1, \cdots, e$ let Y_s denote*(3) $\text{Ann}(P^s, M)$ *and let Y_s' denote* $\text{Ann}(P'^s, M')$. *Then*

$$(3.4) \qquad \text{Ann}(Y_s, M') = M'P'^s;$$

(3) The convention $J^0 = R$ (for ideals J or R) is used.

$$\text{(3.5)} \qquad \operatorname{Ann}(MP^s, M') = Y_s';$$

$$\text{(3.6)} \qquad \operatorname{Ann}(M'P'^s, M) = Y_s;$$

and

$$\text{(3.7)} \qquad \operatorname{Ann}(Y_s', M) = MP^s$$

hold for $s=0, 1, \cdots, e$.

Proof. By the symmetry enjoyed by M and M' only (3.5) and (3.7) require proof. If $z' \in Y_s' = \operatorname{Ann}(P'^s, M')$, $m \in M$, and $\pi \in P^s$, then by the observations preceding this theorem $\pi' \in P'^s$ and $(m\pi, z') = (m, z'\pi') = (m, 0) = 0$. We infer that z' annihilates every element $\sum_1^j m_i\pi_i$ with $\pi_i \in P^s$. Thus $Y_s' \subseteq \operatorname{Ann}(MP^s, M')$. Conversely, if $m' \in \operatorname{Ann}(MP^s, M')$ and $q' \in P'^s$, then $q \in P^s$ and, for all $m \in M$, $(m, m'q') = (mq, m') = 0$. By (3.1) $m'q' = 0$ and $m' \in \operatorname{Ann}(P'^s, M') = Y_s'$, proving that $\operatorname{Ann}(MP^s, M') \subseteq Y_s'$ and completing the proof of (3.5).

We apply (3.5) and the identity [5, p. 27]

$$M_1 = \operatorname{Ann}(\operatorname{Ann}(M_1, M'), M)$$

which holds for subspaces M_1 of M to obtain

$$MP^s = \operatorname{Ann}(\operatorname{Ann}(MP^s, M'), M) = \operatorname{Ann}(Y_s', M)$$

which is (3.7). Q.E.D.

Evidently the subalgebra R is not assumed to be commutative in Theorem 3.2. If R/P is a division ring its dimension over K is less than n^2. Thus its dimension over its center (which contains $\{kI_n + P \mid k \in K\}$) is finite and the division ring R/P is commutative.

Corollary. *Assume now that* $R/P\,[\cong R'/P']$ *is a field. Using the notation of the theorem*

$$\text{(3.8)} \qquad K\text{-}\dim(MP^{s-1}/MP^s) = K\text{-}\dim(Y_s'/Y_{s-1}'), \quad s = 1, 2, \cdots, e;$$

$$\text{(3.9)} \qquad (R/P)\text{-}\dim(M/MP) = (R'/P')\text{-}\dim(\operatorname{Ann}(P', M')).$$

Proof. By (3.5) we have $Y_s' = \operatorname{Ann}(MP^s, M')$. To this we apply a well-known result [5, p. 31]: If M_1 is a subspace of vector space M then M/M_1 and $\operatorname{Ann}(M_1, M')$ are dual vector spaces. Thus Y_s' and M/MP^s are dual spaces and have the same dimension; similar statements hold for Y_{s-1}' and M/MP^{s-1}. Since $\dim Y_s'/Y_{s-1}' = \dim Y_s' - \dim Y_{s-1}'$ and $\dim MP^{s-1}/MP^s = \dim M/MP^s - \dim M/MP^{s-1}$, we have (3.8). Assume now that (3.9) is false: $(R/P)\text{-}\dim M/MP \neq (R'/P')\text{-}\dim(\operatorname{Ann}(P', M'))$. Let t denote $(R/P):K = (R'/P'):K$. From $K\text{-}\dim M/MP = t \cdot (R/P)\text{-}\dim M/MP$; and $K\text{-}\dim(\operatorname{Ann}(P', M')) = t \cdot (R'/P')\text{-}\dim(\operatorname{Ann}(P', M'))$ follows $K\text{-}\dim(M/MP) \neq K\text{-}\dim(\operatorname{Ann}(P', M'))$, a contradiction of the case $s=1$ of (3.8). Q.E.D.

Remark. If A is an arbitrary ring, an A-module V is said to be completely reducible if it can be expressed as a sum of irreducible submodules. A completely reducible module can be expressed as a direct sum of a subfamily of the family of irreducible modules [9, p. 61, Corollary 2 to Theorem 1] the cardinality of which is an invariant [9, p. 62, Theorem 3]. Concerning the maximal completely reducible submodule of an A-module W we have from [2, pp. 103–104, Theorems 9.4A and 9.4C]:

THEOREM A. *If A is a ring satisfying the D.C.C. for right ideals and if W is an A-module, the maximal completely reducible submodule of W is* $\text{Ann}(Q, W)$ *where* $Q = \text{rad } A$.

In the ensuing discussion and theorems in §3 we consider the subalgebra R of K_n to be commutative. We observe that the vector space M satisfies both chain conditions as an R-module since each submodule is a vector space, and R satisfies the chain conditions for ideals by similar reasoning.

THEOREM 3.3. *M is completely indecomposable as an R-module if, and only if, P is a maximal ideal and the (R/P)-dimension of* $\text{Ann}(P, M)$ *is unity.*

Proof. P is a maximal ideal assuming either of the conditions of the Theorem. For Theorem 1.5 implies the maximality of P when M (which satisfies the chain conditions) is indecomposable. Since R satisfies the D.C.C., the completely reducible submodule of M is $\text{Ann}(P, M)$ by Theorem A in the preceding remark. M has at least one irreducible submodule (by the D.C.C.); thus, for some positive integer d, $\text{Ann}(P, M) = \sum_1^d \oplus M_i$, M_i an irreducible submodule for $i = 1, 2, \cdots, d$. We write $M_i = x_i R$, $x_i \in M_i$, since an irreducible module is cyclic [9, p. 6, Proposition 1]. Since $x_i \in \text{Ann}(P, M)$, $x_i P = 0$ and we have

$$\text{Ann}(P, M) = \sum_1^d \oplus x_i R \underset{F}{\cong} \sum_1^d \oplus x_i F$$

where $F \cong R/P$. Thus $d = (R/P)\text{-dim}(\text{Ann}(P, M)) = 1$ if, and only if, M has exactly one irreducible submodule. Since R is a commutative ring with identity and M satisfies the chain conditions, the complete indecomposability of M is, by definition, equivalent to its having exactly one irreducible submodule. Q.E.D.

THEOREM 3.4. *M is a cyclic, indecomposable R-module if, and only if, M' is a completely indecomposable R'-module.*

Proof. By (iii), Theorem 3.1, it follows that M is indecomposable if, and only if, M' is so; consequently, the indecomposability of both modules follows from either of the conditions of this theorem. Since the modules satisfy the chain conditions Theorem 1.5 assures the maximality of $P[\cong P']$. By Theorem 3.2 and corollary $\text{Ann}(P', M')$ and M/MP have the same (R/P)-dimension t. Since the modules are indecomposable and the radical is maximal under either of the conditions of this Theorem, "$t=1$" is equivalent to "M is cyclic" and to "M' is completely indecomposable" (Theorems 1.8 and 3.3), proving the theorem.

REMARK. If M is completely indecomposable for R, M' is cyclic for R' by Theorem 3.4 and R' (hence R also) is maximal commutative in K_n. We have Snapper's Theorem: "$\mathrm{Hom}_R(M, M)=R^*$ if M is completely indecomposable" for the case in which M is an n-dimensional vector space and R a subalgebra of K_n.

DEFINITION OF A CLASS $\mathfrak{M}_n$ OF MAXIMAL COMMUTATIVE SUBALGEBRAS. Let R be commutative and completely primary. The following conditions each of which implies that $\mathrm{Hom}_R(M, M)=R^*$

(i) M is R-cyclic,

(ii) M is an R-module $[C]$,

together with their analogues for the R'-module M', is a set of conditions each of which is sufficient to insure that the completely primary, commutative algebra R is a maximal commutative subalgebra of K_n (Theorem 3.1). The class of maximal commutative, completely primary algebras so obtained will be designated as the class $\mathfrak{M}_n$.

REMARK. Among the conditions known to imply $\mathrm{Hom}_R(M, M)=R^*$ is the condition

(iii) M is completely indecomposable in the sense of Snapper.

Since by Theorem 3.4 (iii) holds for $M[M']$ only if (i) holds for $M'[M]$, it is not necessary to include (iii) in the definition of the class $\mathfrak{M}_n$.

THEOREM 3.5. *If M is a cyclic or completely indecomposable R-module, then K-dim $R=n$.*

Proof. If $M=xR$, $x\in M$, the mapping $r\rightarrow xr$ of R onto M is an R-isomorphism, since M is faithful. The isomorphism is a K-isomorphism since $KI_n\subseteq R$, and we conclude that R and M have the same dimensionality. If M is completely indecomposable for R then M' is R'-cyclic (Theorem 3.4) and we have dim $R=$dim $R'=n$ from the first case of this theorem.

REMARK. If R is completely primary and if $R/P\cong K$ then, by Theorem 3.2 and corollary and Theorem 1.8, K-dim(Ann(P, M)) $=K$-dim$(M'/M'P')$ $=1$ if M' is R'-cyclic, and K-dim $M/MP=1$ if M is R-cyclic. In either case, if $k=K$-dim(M/MP) and $d=K$-dim(Ann(P, M)), $(k-1)(d-1)=0$. By Theorem 3.5, K-dim $M=K$-dim R when M or M' is cyclic, and the formula

$$K\text{-dim } R - K\text{-dim } M = (k-1)(d-1) \tag{3.10}$$

holds.

THEOREM 3.6. *If $R\in\mathfrak{M}_n$ is such that $R/P\cong K$ and if k and d denote the dimensions of M/MP and* Ann(P, M) *respectively, then* (3.10) *holds.*

Proof. Because of the preceding remark (3.10) needs proof only for the cases in which M is an R-module $[C]$ or M' is an R'-module $[C]$. If k' denotes $\dim M'/M'P'$ and d' denotes $\dim(\mathrm{Ann}(P', M'))$, $d=k'$ and $d'=k$ by (3.9), corollary to Theorem 3.2. Since $\dim M=\dim M'$ and $\dim R=\dim R'$, (3.10) holds if, and only if, $\dim R'-\dim M'=(k'-1)(d'-1)$. Thus it is sufficient to prove (3.10) assuming that M is an R-module $[C]$. By hypothesis the dimension of R/P is unity. By Theorem 2.7, $\mathrm{Hom}_R(M, M)=R^*$, and we have from conclusion (4), Theorem 2.3: $\dim(\mathrm{Ann}(MP, R))=kd$. Thus $\dim R=1+kd+\dim\{P/\mathrm{Ann}(MP, R)\}$, and (3.10) is equivalent to $1+kd+\dim\{P/\mathrm{Ann}(MP, R)\}-n=(k-1)(d-1)$, or

$$(3.11)\qquad \dim\{P/\mathrm{Ann}(MP, R)\} = n - k - d.$$

By definitions of k and d the right member of (3.11) is the dimension of $MP/\mathrm{Ann}(P, M)$. We prove (3.11) by showing that

$$(3.12)\qquad P/\mathrm{Ann}(MP, R) \underset{K}{\cong} MP/\mathrm{Ann}(P, M).$$

From the definition of "R-module $[C]$" $M=\sum_1^k x_iR$, $x_i\in M$, and by Theorem 2.8

$$(3.13)\qquad MP/\mathrm{Ann}(P, M) \cong \bar{x}_1P \oplus \cdots \oplus \bar{x}_kP \quad \text{where } \bar{v} \text{ denotes } v+\mathrm{Ann}(P, M).$$

We let $x=\sum_1^k x_i$ and consider the mapping E of P into $MP/\mathrm{Ann}(P, M)$ defined by

$$E(p) = \bar{x}p, \qquad p \in P.$$

If $p_1, p_2, \cdots, p_k$ are elements of P, then by (C_2''), Lemma 2.4, there exists an element $p\in P$ such that $p\equiv p_i \bmod \mathrm{Ann}(x_i, R)$, $1\leqq i\leqq k$; we have $xp=\sum_1^k x_ip=\sum_1^k x_ip_i$. Considering (3.13) E is a mapping onto $MP/\mathrm{Ann}(P, M)$.

Since $K\subseteq R$ the R-homomorphism E is a K-homomorphism. Now $\ker E=\{p\mid \bar{x}p=\bar{0}\}=\{p\mid \sum_1^k x_ip\in \mathrm{Ann}(P, M)\}=\{p\mid x_ip\in\mathrm{Ann}(P, M),\ 1\leqq i\leqq k\}$. The last equality comes from the direct sum in (3.13). Thus $\ker E=\{p\mid mp\in\mathrm{Ann}(P, M) \text{ for all } m\}$, since $M=\sum x_iR$, and we have $\ker E=\mathrm{Ann}(MP, R)$, proving (3.12) and the theorem.

REMARK. Since R/P is a finite extension of a field isomorphic to K, K and R/P will be isomorphic if K is algebraically closed. Thus Theorem 3.6 is applicable to all algebras in $\mathfrak{M}_n$ provided the field K is algebraically closed.

4. **Examples. Scope of the class $\mathfrak{M}_n$.** Demonstration that a commutative, completely primary subalgebra R of K_n is maximal commutative is usually greatly simplified when the representation module of R (or of the algebra R' of adjoints of R) is an R-module $[C]$ or is cyclic. That these conditions need not exist for a maximal commutative subalgebra is made clear in Example 4.3. First we present an example of an algebra whose representation module is an R-module $[C]$ but is neither a cyclic nor a completely indecomposable R-module.

In what follows the unit matrices in K_n will be denoted by E_{ij} $(1 \leq i, j \leq n)$ and the identity matrix by I_n.

EXAMPLE 4.1. Let M be a 6-dimensional vector space $Ku_1 \oplus \cdots \oplus Ku_6$ acted upon by

$$R = \left\| \begin{matrix} a & b & c & 0 & d & e \\ 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & f & a & g & h \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{matrix} \right\| \qquad a, b, c, d, e, f, g, h \text{ arbitrary in } K.$$

As a vector space R has a basis consisting of I_6 and the set

$$S = \{E_{12} + E_{23}, E_{13}, E_{15}, E_{16}, E_{43}, E_{45}, E_{46}\}.$$

The product xy, $(x, y) \in S \times S$, is zero unless $x = y = E_{12} + E_{23}$, and $(E_{12} + E_{23})^2 = E_{13}$. Thus, if P denotes the space generated by S, $P^3 = 0$, and multiplication is closed and commutative in P (and clearly is so in R). P is an ideal since $P \supseteq PI_6$. Considering the dimensions of P and R, P is a nilpotent, maximal ideal, proving that P is the radical of R. Since $MP = Ku_2 \oplus Ku_3 \oplus Ku_5 \oplus Ku_6$, $\dim M/MP = 2$, proving that M is not R-cyclic (Theorem 1.8). For $i = 3, 5, 6$, we verify $Ku_i = u_iR$ and $u_iP = 0$ and conclude that $\mathrm{Ann}(P, M)$ contains the independent irreducible R-modules Ku_3, Ku_5, and Ku_6. This proves that M is not completely indecomposable (Theorem 3.3). If $v = \sum c_iu_i \in \mathrm{Ann}(P, M)$ then $v(E_{12} + E_{23} + E_{45}) = c_1u_2 + c_2u_3 + c_4u_5 = 0$ implies $c_1 = c_2 = c_4 = 0$, proving that $v \in Ku_3 \oplus Ku_5 \oplus Ku_6$. Thus $\mathrm{Ann}(P, M) = Ku_3 \oplus Ku_5 \oplus Ku_6$.

Since $u_i \in u_1R$ $(i = 1, 2, 3, 5, 6)$ and $u_4 \in u_4R$, $M = u_1R + u_4R$. We claim that $\{u_1 + MP, u_4 + MP\}$ is a basis of M/MP in which the requirements (C_1) and (C_2) for an R-module $[C]$ are satisfied. The equations $u_3 = u_1E_{13} = u_4E_{43}$, $u_5 = u_1E_{15} = u_4E_{45}$, and $u_6 = u_1E_{16} = u_4E_{46}$ prove $\mathrm{Ann}(P, M) \subseteq u_iP$ for $i = 1, 4$, which is (C_1). To obtain (C_2): $P = \mathrm{Ann}(u_1, R) + \mathrm{Ann}(u_4, R)$ one verifies that $\mathrm{Ann}(u_1, R) \supseteq KE_{43} \oplus KE_{45} \oplus KE_{46}$ and that $\mathrm{Ann}(u_4, R) \supseteq K(E_{12} + E_{23}) \oplus KE_{13} \oplus KE_{15} \oplus KE_{16}$. M, then, is an R-module $[C]$ and R is maximal commutative in K_6 by Theorem 2.7.

EXAMPLE 4.2. (An R-module $[C]$ that satisfies neither the A.C.C. nor the D.C.C.) Let M be an infinite-dimensional vector space $Ku_1 \oplus Ku_2 \oplus \cdots$ and let L denote the ring of linear transformations on M or, equivalently, of row-finite infinite matrices. The set $S = \{E_{ij} \mid i = 1, 2; j = 3, 4, \cdots\}$ generates a zero subalgebra P of L. Then the subalgebra R generated by S and the identity transformation is commutative (since P is a zero algebra) and its radical is the niltpotent, maximal ideal P. Direct verification yields

$$M = u_1R + u_2R, \tag{1}$$

$$MP = \mathrm{Ann}(P, M) = \sum_{i>2} u_iR \subseteq u_jP, \qquad j = 1, 2. \tag{2}$$

By (2) the basis $\{u_1+MP, u_2+MP\}$ of M/MP satisfies (C_1). Ann(u_1, R) contains $\{E_{2j}|j=3, 4, \cdots\}$ and Ann(u_2, R) contains $\{E_{1j}|j=3, 4, \cdots\}$, so that $S\subseteq$Ann(u_1, R)+Ann(u_2, R). Since Ann(u_i, R)$\subseteq P$ by the (R/P)-independence of $\{u_i|i=1, 2\}$ and since S generates P, we have

$$(C_2): \qquad P = \text{Ann}(u_1, R) + \text{Ann}(u_2, R).$$

M, then, is an R-module $[C]$, and R is maximal commutative in L. The A.C.C. and the D.C.C. fail to hold in M since Ann(P, M) contains the infinite set of independent submodules $\{Ku_i|i=3, 4, \cdots\}$.

The remainder of this paper will be devoted to showing that the class of maximal commutative, completely primary subalgebras of K_n is not exhausted by the class $\mathfrak{M}_n$ defined in §3. The counter-example (Example 4.3) is preceded by two theorems.

THEOREM 4.1. *Let K be a field and for $i=1, 2$, let n_i and e_i be positive integers. For $i=1, 2$, let R_i be a commutative, completely primary subalgebra of K_{n_i} with radical P_i such that e_i is the exponent of P_i and $R_i/P_i\cong K$. Let R denote the Kronecker product ring $R_1\otimes_K R_2$ and let P denote* rad R. *Then $P=(R_1\otimes_K P_2)+(P_1\otimes_K R_2)$, e_1+e_2-1 is the exponent e of P, and $R/P\cong K$. Thus R is completely primary.*

Proof. If Q denotes $(R_1\otimes P_2)+(P_1\otimes R_2)$, Q is clearly an ideal in R. If, for $i=1, 2, \cdots, e_1+e_2-1$, $w_i=g_i\otimes h_i\in Q$, either e_1 of the g_i belong to P_1 or e_2 of the h_i belong to P_2, so that $\prod_{i=1}^{e_1+e_2-1} w_i=0$; the same conclusion is reached if each w_i has the form $\sum_j g_{ij}\otimes h_{ij}$, proving that $Q^{e_1+e_2-1}=0$. Let m_i denote dim R_i, $i=1, 2$, and let $\{u_1=I_{n_1}, u_2, \cdots, u_{m_1}\}$ and $\{v_1=I_{n_2}, v_2, \cdots, v_{m_2}\}$ be bases of R_1 and R_2, respectively, such that $u_i\in P_1$ and $v_i\in P_2$ for $i>1$. Then $I_{n_1n_2}$ and the set S of m_1m_2-1 independent elements $\{u_i\times v_j|i+j>2\}$ form a basis of R. One verifies $S\subseteq Q$, and (since Q is nilpotent) $I_{n_1n_2}\notin Q$. Thus dim $Q=m_1m_2-1$ and Q is a nilpotent, maximal ideal in the (m_1m_2)-dimensional algebra R. P, then, is necessarily Q and has exponent $e\leqq e_1+e_2-1$. To prove $e=e_1+e_2-1$, we observe that if s and t are nonzero elements of $P_1^{e_1-1}$ and $P_2^{e_2-1}$, respectively, then $s\otimes t[=(s\otimes I_{n_2})\cdot(I_{n_1}\otimes t)]$ is a nonzero element of $P^{e_1+e_2-2}$. The isomorphism between R/P and K is clear since dim P $=$dim $R-1$.

NOTATION. If R is a ring and S is a subset of R, we denote by R^S the set of elements of R that commute with every element of S.

THEOREM 4.2. *Let A and B be rings with identity element whose centers contain a field K. Let $U\subseteq A$ and $V\subseteq B$ be rings containing K. Then $(A\otimes_K B)^{U\otimes_K V} = A^U\otimes_K B^V$.*

A proof of Theorem 4.2 appears in [2, p. 68, Lemma 7.3B].

COROLLARY. *If $U[V]$ is a maximal commutative subalgebra of $K_{n_1}[K_{n_2}]$ then $U\otimes_K V$ is isomorphic to a maximal commutative subalgebra of $K_{n_1n_2}$.*

Proof. It is proved elsewhere [7, p. 226, Theorem 6] that $K_{n_1} \otimes K_{n_2} \cong K_{n_1 n_2}$ as algebras. We shall have the result, then, if $U \otimes V$ is maximal commutative in $K_{n_1} \otimes K_{n_2}$. By hypothesis $U = K_{n_1}^U$ and $V = K_{n_2}^V$, and by the theorem $(K_{n_1} \otimes K_{n_2})^{(U \otimes V)} = K_{n_1}^U \otimes K_{n_2}^V = U \otimes V$.

Toward constructing Example 4.3, let R_1 denote the subalgebra of K_3 which consists of matrices of the form

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

where a, b, and c are arbitrary elements of K. R_1 as a vector space over K is generated by I_3, E_{12}, and E_{13} and, since $E_{12}^2 = E_{13}^2 = E_{12}E_{13} = E_{13}E_{12} = 0$, multiplication in R_1 is closed and commutative. Since the nilpotent, maximal ideal $KE_{12} \oplus KE_{13}$ is the radical P_1 of R_1, R_1/P_1 is isomorphic to K. If $M = Ku_1 \oplus Ku_2 \oplus Ku_3$ denotes the representation module of R_1 then, since $M[=u_1R_1]$ is cyclic, R_1 is a maximal commutative subalgebra of K_3. The algebra R_2 of adjoints of R_1 is likewise a maximal commutative, completely primary subalgebra of K_3 with $R_2/P_2 \cong K$. R_2 is the algebra of all matrices in the form

$$\begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix}$$

where a, b, and c are elements of K.

Example 4.3. (A completely primary, maximal commutative subalgebra R of K_9 which is not in the class $\mathfrak{M}_9$). Let $R = R_1 \otimes R_2$. By Theorems 4.1 and 4.2 with corollary, R is a maximal commutative, completely primary subalgebra of K_9 with $R/P \cong K$. We shall see that R does not satisfy the conclusion of Theorem 3.6 and consequently is not in the class $\mathfrak{M}_9$. The matrix representation of R is

$$(4.1) \quad R = \left(\begin{array}{ccc:ccc:ccc} a & 0 & 0 & d & 0 & 0 & g & 0 & 0 \\ b & a & 0 & e & d & 0 & h & g & 0 \\ c & 0 & a & f & 0 & d & k & 0 & g \\ \hdashline & & & a & 0 & 0 & & & \\ & 0 & & b & a & 0 & & 0 & \\ & & & c & 0 & a & & & \\ \hdashline & & & & & & a & 0 & 0 \\ & 0 & & & 0 & & b & a & 0 \\ & & & & & & c & 0 & a \end{array}\right), \qquad a, b, c, d, e, f, g, h, k \in K.$$

The matrix (4.1) conforms with the schema of the Kronecker product of two matrices appearing in [7, p. 213]. By the same schema the matrix (4.1) with $a=0$ reveals the 8-dimensional algebra $P_1\otimes R_2+R_1\otimes P_2$ which by Theorem 4.1 is P[=rad R] and has exponent 3 [=exponent of P_1+exponent of P_2-1]. Translated into matrix units P has the basis

$$\{E_{31}+E_{64}+E_{97}, E_{21}+E_{54}+E_{87}, E_{14}+E_{25}+E_{36}, \\ E_{17}+E_{28}+E_{39}, E_{24}, E_{34}, E_{27}, E_{37}\}. \tag{4.2}$$

According to Theorem 3.6, if R is in the class $\mathfrak{M}_9$ defined in §3, the following equation must be satisfied

$$\dim R - \dim M = (k-1)(d-1) \tag{4.3}$$

where $M[=Kw_1\oplus \cdots \oplus Kw_9]$ is the space on which R acts, $k=\dim(M/MP)$, and $d=\dim(\mathrm{Ann}(P, M))$. Examination of the submodules Mp, p a generator (see (4.2)) of P, discloses that $MP=Kw_1\oplus Kw_4\oplus Kw_5\oplus Kw_6\oplus Kw_7\oplus Kw_8\oplus Kw_9$, whence $k=2$. The list (4.2) discloses also that $Kw_4\oplus Kw_7\subseteq\mathrm{Ann}(P, M)$. If $x=\sum c_iw_i\in\mathrm{Ann}(P, M)$, the equations $0=x(E_{31}+E_{64}+E_{97})=c_3w_1+c_6w_4+c_9w_7$, $0=x(E_{21}+E_{54}+E_{87})=c_2w_1+c_5w_4+c_8w_7$, and $0=x(E_{14}+E_{25}+E_{36})=c_1w_4+c_2w_5+c_3w_6$ imply that $c_1=c_2=c_3=c_5=c_6=c_8=c_9=0$, so that $x=c_4w_4+c_7w_7$. Thus $\mathrm{Ann}(P, M)=Kw_4\oplus Kw_7$ and $\dim(\mathrm{Ann}(P, M))=2$. Now the left-hand side of (4.3) is 0 and the right-hand side is 1, proving that R is not in the class $\mathfrak{M}_9$. Thus it is demonstrated that the aggregate of the conditions discussed in this paper which imply that a commutative, completely primary algebra is maximal commutative in K_n does not constitute a necessary condition.

It is worth mentioning that the failure to get a necessary condition occurred at exponent $e=3$. According to conclusions (3) and (4′) of Theorem 2.3 M is necessarily an R-module [C] if $\mathrm{Hom}_R(M, M)=R^*$ under somewhat general hypotheses including $e=2$.

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